# Exercise 2.5.15

Solve Laplace's equation inside a semi-infinite strip ($0 < x < \infty$, $0 < y < H$) subject to the boundary conditions [*Hint*: In Cartesian coordinates, $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, inside a semi-infinite strip ($0 \le y \le H$ and $0 \le x < \infty$), it is known that if $u(x,y) = F(x)G(y)$, then $\frac{1}{F}\frac{d^2F}{dx^2} = -\frac{1}{G}\frac{d^2G}{dy^2}$.]:

**(a)** $\frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,H) = 0, \quad u(0,y) = f(y)$

**(b)** $u(x,0) = 0, \quad u(x,H) = 0, \quad u(0,y) = f(y)$

**(c)** $u(x,0) = 0, \quad u(x,H) = 0, \quad \frac{\partial u}{\partial x}(0,y) = f(y)$

**(d)** $\frac{\partial u}{\partial y}(x,0) = 0, \quad \frac{\partial u}{\partial y}(x,H) = 0, \quad \frac{\partial u}{\partial x}(0,y) = f(y)$

Show that the solution [part (d)] exists only if $\int_0^H f(y)\,dy = 0$.

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### Solution

Here the Laplace equation will be solved in a semi-infinite rectangular domain. Because the Laplace equation is linear and homogeneous, it can be solved with the method of separation of variables.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Assume a product solution of the form $u(x,y) = X(x)Y(y)$ and plug it into the PDE.

$$\frac{\partial^2}{\partial x^2}[X(x)Y(y)] + \frac{\partial^2}{\partial y^2}[X(x)Y(y)] = 0$$

$$X''Y + XY'' = 0$$

Divide both sides by $X(x)Y(y)$.

$$\frac{X''}{X} + \frac{Y''}{Y} = 0$$

$$\frac{X''}{X} = -\frac{Y''}{Y}$$

The only way a function of $x$ can be equal to a function of $y$ is if both are equal to a constant $\lambda$.

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda$$

As a result of separating variables, the PDE has reduced to two ODEs—one in each independent variable.

$$\left.\begin{aligned} \frac{X''}{X} &= \lambda \\ -\frac{Y''}{Y} &= \lambda \end{aligned}\right\}$$

Values of $\lambda$ for which nontrivial solutions to these ODEs and the associated boundary conditions exist are called eigenvalues, and the solutions themselves are called eigenfunctions. Note that it doesn't matter what side the minus sign is on as long as all eigenvalues are considered.

## Part (a)

Substitute the product solution into the homogeneous boundary conditions.

$$\begin{aligned}
\frac{\partial u}{\partial y}(x,0) = 0 \quad &\rightarrow \quad X(x)Y'(0) = 0 \quad \rightarrow \quad Y'(0) = 0 \\
\frac{\partial u}{\partial y}(x,H) = 0 \quad &\rightarrow \quad X(x)Y'(H) = 0 \quad \rightarrow \quad Y'(H) = 0
\end{aligned}$$

Solve the ODE for $Y$.

$$Y'' = -\lambda Y$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y$$

Differentiate it with respect to $y$.

$$Y'(y) = \mu(-C_1 \sin \mu y + C_2 \cos \mu y)$$

Apply the boundary conditions to determine $C_1$ and $C_2$.

$$\begin{aligned}
Y'(0) &= \mu(C_2) = 0 \\
Y'(H) &= \mu(-C_1 \sin \mu H + C_2 \cos \mu H) = 0
\end{aligned}$$

Since $C_2 = 0$, the second equation reduces to $-C_1 \mu \sin \mu H = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\begin{aligned}
\sin \mu H &= 0 \\
\mu H &= n\pi, \quad n = 1, 2, \ldots \\
\mu &= \frac{n\pi}{H}
\end{aligned}$$

There are positive eigenvalues $\lambda = \left(\frac{n\pi}{H}\right)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y \quad \rightarrow \quad Y_n(y) = \cos \frac{n\pi y}{H}.$$

Only positive integers are taken for $n$ because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values for $\lambda$. Using $\lambda = \frac{n^2\pi^2}{H^2}$, solve the ODE for $X$ now.

$$X'' = \frac{n^2\pi^2}{H^2} X$$

The general solution can be written in terms of exponential functions.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right) + B \exp\left(\frac{n\pi}{H}x\right)$$

In order to prevent the solution from blowing up as $x \to \infty$, set $B = 0$.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right)$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$Y'' = 0$$

The general solution is a straight line.

$$Y(y) = C_3 y + C_4$$

Differentiate it with respect to $y$.

$$Y'(y) = C_3$$

Apply the boundary conditions to determine $C_3$.

$$Y'(0) = C_3 = 0$$
$$Y'(H) = C_3 = 0$$

$C_4$ remains arbitrary.

$$Y(y) = C_4$$

This is not the trivial solution, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for $X$.

$$X'' = 0$$

The general solution is a straight line.

$$X(x) = Dx + E$$

In order to prevent the solution from blowing up as $x \to \infty$, set $D = 0$.

$$X(x) = E$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_5 \cosh \gamma y + C_6 \sinh \gamma y$$

Differentiate it with respect to $y$.

$$Y'(y) = \gamma(C_5 \sinh \gamma y + C_6 \cosh \gamma y)$$

Apply the boundary conditions to determine $C_5$ and $C_6$.

$$Y'(0) = \gamma(C_6) = 0$$
$$Y'(H) = \gamma(C_5 \sinh \gamma H + C_6 \cosh \gamma H) = 0$$

Since $C_6 = 0$, the second equation reduces to $C_5 \gamma \sinh \gamma H = 0$. No nonzero value of $\gamma$ can satisfy this equation, so $C_5 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so there are no negative eigenvalues. According to the principle of superposition, the solution to the PDE is a linear combination of the eigenfunctions $u = X_n(x)Y_n(x)$ over all the eigenvalues.

$$\boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n\pi}{H}x\right)\cos\frac{n\pi y}{H}}$$

Use the final boundary condition to determine the coefficients, $A_0$ and $A_n$.

$$u(0,y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi y}{H} = f(y) \tag{1}$$

Integrate both sides with respect to $y$ from 0 to $H$ to get $A_0$.

$$\int_0^H \left(A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi y}{H}\right) dy = \int_0^H f(y)\,dy$$

Split up the integral on the left and bring the constants in front.

$$A_0 \underbrace{\int_0^H dy}_{=H} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^H \cos\frac{n\pi y}{H}\,dy}_{=0} = \int_0^H f(y)\,dy$$

Evaluate the integrals.

$$A_0 H = \int_0^H f(y)\,dy$$

Therefore,

$$\boxed{A_0 = \frac{1}{H}\int_0^H f(y)\,dy.}$$

To get $A_n$, multiply both sides of equation (1) by $\cos\frac{p\pi y}{H}$, where $p$ is an integer.

$$A_0\cos\frac{p\pi y}{H} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi y}{H}\cos\frac{p\pi y}{H} = f(y)\cos\frac{p\pi y}{H}$$

Integrate both sides with respect to $y$ from 0 to $H$.

$$\int_0^H \left(A_0\cos\frac{p\pi y}{H} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi y}{H}\cos\frac{p\pi y}{H}\right) dy = \int_0^H f(y)\cos\frac{p\pi y}{H}\,dy$$

Split up the integrals and bring the constants in front.

$$A_0 \underbrace{\int_0^H \cos\frac{p\pi y}{H}\,dy}_{=0} + \sum_{n=1}^{\infty} A_n \int_0^H \cos\frac{n\pi y}{H}\cos\frac{p\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{p\pi y}{H}\,dy$$

Because the cosine functions are orthogonal, this second integral on the left is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$A_n \int_0^H \cos^2\frac{n\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{n\pi y}{H}\,dy$$

Evaluate the integral.

$$A_n\left(\frac{H}{2}\right) = \int_0^H f(y)\cos\frac{n\pi y}{H}\,dy$$

Therefore,

$$\boxed{A_n = \frac{2}{H}\int_0^H f(y)\cos\frac{n\pi y}{H}\,dy.}$$

## Part (b)

Substitute the product solution into the homogeneous boundary conditions.

$$\begin{aligned} u(x,0) = 0 \quad &\rightarrow \quad X(x)Y(0) = 0 \quad &\rightarrow \quad Y(0) = 0 \\ u(x,H) = 0 \quad &\rightarrow \quad X(x)Y(H) = 0 \quad &\rightarrow \quad Y(H) = 0 \end{aligned}$$

Solve the ODE for $Y$.

$$Y'' = -\lambda Y$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1\cos\mu y + C_2\sin\mu y$$

Apply the boundary conditions to determine $C_1$ and $C_2$.

$$\begin{aligned} Y(0) &= C_1 = 0 \\ Y(H) &= C_1\cos\mu H + C_2\sin\mu H = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2\sin\mu H = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$.

$$\begin{aligned} \sin\mu H &= 0 \\ \mu H &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= \frac{n\pi}{H} \end{aligned}$$

There are positive eigenvalues $\lambda = \left(\frac{n\pi}{H}\right)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_1\cos\mu y + C_2\sin\mu y \quad \rightarrow \quad Y_n(y) = \sin\frac{n\pi y}{H}.$$

Only positive integers are taken for $n$ because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values for $\lambda$. Using $\lambda = \frac{n^2\pi^2}{H^2}$, solve the ODE for $X$ now.

$$X'' = \frac{n^2\pi^2}{H^2}X$$

The general solution can be written in terms of exponential functions.

$$X(x) = A\exp\left(-\frac{n\pi}{H}x\right) + B\exp\left(\frac{n\pi}{H}x\right)$$

In order to prevent the solution from blowing up as $x \to \infty$, set $B = 0$.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right)$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$Y'' = 0$$

The general solution is a straight line.

$$Y(y) = C_3 y + C_4$$

Apply the boundary conditions to determine $C_3$ and $C_4$.

$$\begin{aligned} Y(0) &= C_4 = 0 \\ Y(H) &= C_3 H + C_4 = 0 \end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 H = 0$, which means $C_3 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_5 \cosh \gamma y + C_6 \sinh \gamma y$$

Apply the boundary conditions to determine $C_5$ and $C_6$.

$$\begin{aligned} Y(0) &= C_5 = 0 \\ Y(H) &= C_5 \cosh \gamma H + C_6 \sinh \gamma H = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sinh \gamma H = 0$. No nonzero value of $\gamma$ can satisfy this equation, so $C_6 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so there are no negative eigenvalues. According to the principle of superposition, the solution to the PDE is a linear combination of the eigenfunctions $u = X_n(x)Y_n(x)$ over all the eigenvalues.

$$\boxed{u(x, y) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n\pi}{H}x\right) \sin \frac{n\pi y}{H}}$$

Use the final boundary condition to determine the coefficients $A_n$.

$$u(0, y) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} = f(y)$$

Multiply both sides by $\sin \frac{p\pi y}{H}$, where $p$ is an integer.

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H} = f(y) \sin \frac{p\pi y}{H}$$

Integrate both sides with respect to $y$ from 0 to $H$.

$$\int_0^H \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H}\, dy = \int_0^H f(y) \sin \frac{p\pi y}{H}\, dy$$

Split up the integrals and bring the constants in front.

$$\sum_{n=1}^{\infty} A_n \int_0^H \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H}\, dy = \int_0^H f(y) \sin \frac{p\pi y}{H}\, dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$A_n \int_0^H \sin^2 \frac{n\pi y}{H}\, dy = \int_0^H f(y) \sin \frac{n\pi y}{H}\, dy$$

Evaluate the integral.

$$A_n \left(\frac{H}{2}\right) = \int_0^H f(y) \sin \frac{n\pi y}{H}\, dy$$

Therefore,

$$\boxed{A_n = \frac{2}{H} \int_0^H f(y) \sin \frac{n\pi y}{H}\, dy.}$$

**Part (c)**

Substitute the product solution into the homogeneous boundary conditions.

$$\begin{array}{lllll} u(x,0) = 0 & \rightarrow & X(x)Y(0) = 0 & \rightarrow & Y(0) = 0 \\ u(x,H) = 0 & \rightarrow & X(x)Y(H) = 0 & \rightarrow & Y(H) = 0 \end{array}$$

Solve the ODE for $Y$.

$$Y'' = -\lambda Y$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y$$

Apply the boundary conditions to determine $C_1$ and $C_2$.

$$\begin{aligned} Y(0) &= C_1 = 0 \\ Y(H) &= C_1 \cos \mu H + C_2 \sin \mu H = 0 \end{aligned}$$

Since $C_1 = 0$, the second equation reduces to $C_2 \sin \mu H = 0$. To avoid the trivial solution, we insist that $C_2 \neq 0$.

$$\begin{aligned} \sin \mu H &= 0 \\ \mu H &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= \frac{n\pi}{H} \end{aligned}$$

There are positive eigenvalues $\lambda = \left(\frac{n\pi}{H}\right)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y \quad \to \quad Y_n(y) = \sin \frac{n\pi y}{H}.$$

Only positive integers are taken for $n$ because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values for $\lambda$. Using $\lambda = \frac{n^2\pi^2}{H^2}$, solve the ODE for $X$ now.

$$X'' = \frac{n^2\pi^2}{H^2} X$$

The general solution can be written in terms of exponential functions.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right) + B \exp\left(\frac{n\pi}{H}x\right)$$

In order to prevent the solution from blowing up as $x \to \infty$, set $B = 0$.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right)$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$Y'' = 0$$

The general solution is a straight line.

$$Y(y) = C_3 y + C_4$$

Apply the boundary conditions to determine $C_3$ and $C_4$.

$$\begin{aligned} Y(0) &= C_4 = 0 \\ Y(H) &= C_3 H + C_4 = 0 \end{aligned}$$

Since $C_4 = 0$, the second equation reduces to $C_3 H = 0$, which means $C_3 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so zero is not an eigenvalue. Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_5 \cosh \gamma y + C_6 \sinh \gamma y$$

Apply the boundary conditions to determine $C_5$ and $C_6$.

$$\begin{aligned} Y(0) &= C_5 = 0 \\ Y(H) &= C_5 \cosh \gamma H + C_6 \sinh \gamma H = 0 \end{aligned}$$

Since $C_5 = 0$, the second equation reduces to $C_6 \sinh \gamma H = 0$. No nonzero value of $\gamma$ can satisfy this equation, so $C_6 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so there are no negative eigenvalues. According to the principle of superposition, the solution to the PDE is a linear combination of the eigenfunctions $u = X_n(x)Y_n(x)$ over all the eigenvalues.

$$\boxed{u(x,y) = \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n\pi}{H}x\right) \sin \frac{n\pi y}{H}}$$

Differentiate it with respect to $x$.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n \left(-\frac{n\pi}{H}\right) \exp\left(-\frac{n\pi}{H}x\right) \sin \frac{n\pi y}{H}$$

Use the final boundary condition to determine the coefficients $A_n$.

$$\frac{\partial u}{\partial x}(0,y) = \sum_{n=1}^{\infty} A_n \left(-\frac{n\pi}{H}\right) \sin \frac{n\pi y}{H} = f(y)$$

Multiply both sides by $\sin \frac{p\pi y}{H}$, where $p$ is an integer.

$$\sum_{n=1}^{\infty} A_n \left(-\frac{n\pi}{H}\right) \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H} = f(y) \sin \frac{p\pi y}{H}$$

Integrate both sides with respect to $y$ from 0 to $H$.

$$\int_0^H \sum_{n=1}^{\infty} A_n \left(-\frac{n\pi}{H}\right) \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H}\, dy = \int_0^H f(y) \sin \frac{p\pi y}{H}\, dy$$

Split up the integrals and bring the constants in front.

$$\sum_{n=1}^{\infty} A_n \left(-\frac{n\pi}{H}\right) \int_0^H \sin \frac{n\pi y}{H} \sin \frac{p\pi y}{H}\, dy = \int_0^H f(y) \sin \frac{p\pi y}{H}\, dy$$

Because the sine functions are orthogonal, the integral on the left is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$A_n \left(-\frac{n\pi}{H}\right) \int_0^H \sin^2 \frac{n\pi y}{H}\, dy = \int_0^H f(y) \sin \frac{n\pi y}{H}\, dy$$

Evaluate the integral.

$$A_n \left(-\frac{n\pi}{H}\right) \left(\frac{H}{2}\right) = \int_0^H f(y) \sin \frac{n\pi y}{H}\, dy$$

Therefore,

$$\boxed{A_n = -\frac{2}{n\pi} \int_0^H f(y) \sin \frac{n\pi y}{H}\, dy.}$$

**Part (d)**

Substitute the product solution into the homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial u}{\partial y}(x,0) = 0 \quad &\rightarrow \quad X(x)Y'(0) = 0 \quad \rightarrow \quad Y'(0) = 0 \\ \frac{\partial u}{\partial y}(x,H) = 0 \quad &\rightarrow \quad X(x)Y'(H) = 0 \quad \rightarrow \quad Y'(H) = 0 \end{aligned}$$

Solve the ODE for $Y$.

$$Y'' = -\lambda Y$$

Check to see if there are positive eigenvalues: $\lambda = \mu^2$.

$$Y'' = -\mu^2 Y$$

The general solution can be written in terms of sine and cosine.

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y$$

Differentiate it with respect to $y$.

$$Y'(y) = \mu(-C_1 \sin \mu y + C_2 \cos \mu y)$$

Apply the boundary conditions to determine $C_1$ and $C_2$.

$$\begin{aligned} Y'(0) &= \mu(C_2) = 0 \\ Y'(H) &= \mu(-C_1 \sin \mu H + C_2 \cos \mu H) = 0 \end{aligned}$$

Since $C_2 = 0$, the second equation reduces to $-C_1\mu \sin \mu H = 0$. To avoid the trivial solution, we insist that $C_1 \neq 0$.

$$\begin{aligned} \sin \mu H &= 0 \\ \mu H &= n\pi, \quad n = 1, 2, \ldots \\ \mu &= \frac{n\pi}{H} \end{aligned}$$

There are positive eigenvalues $\lambda = \left(\frac{n\pi}{H}\right)^2$, and the eigenfunctions associated with them are

$$Y(y) = C_1 \cos \mu y + C_2 \sin \mu y \quad \rightarrow \quad Y_n(y) = \cos \frac{n\pi y}{H}.$$

Only positive integers are taken for $n$ because $n = 0$ leads to the zero eigenvalue, and negative integers lead to redundant values for $\lambda$. Using $\lambda = \frac{n^2\pi^2}{H^2}$, solve the ODE for $X$ now.

$$X'' = \frac{n^2\pi^2}{H^2} X$$

The general solution can be written in terms of exponential functions.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right) + B \exp\left(\frac{n\pi}{H}x\right)$$

In order to prevent the solution from blowing up as $x \to \infty$, set $B = 0$.

$$X(x) = A \exp\left(-\frac{n\pi}{H}x\right)$$

Check to see if zero is an eigenvalue: $\lambda = 0$.

$$Y'' = 0$$

The general solution is a straight line.

$$Y(y) = C_3 y + C_4$$

Differentiate it with respect to $y$.

$$Y'(y) = C_3$$

Apply the boundary conditions to determine $C_3$.

$$\begin{aligned} Y'(0) &= C_3 = 0 \\ Y'(H) &= C_3 = 0 \end{aligned}$$

$C_4$ remains arbitrary.

$$Y(y) = C_4$$

This is not the trivial solution, so zero is an eigenvalue. Using $\lambda = 0$, solve the ODE for $X$.

$$X'' = 0$$

The general solution is a straight line.

$$X(x) = Dx + E$$

In order to prevent the solution from blowing up as $x \to \infty$, set $D = 0$.

$$X(x) = E$$

Check to see if there are negative eigenvalues: $\lambda = -\gamma^2$.

$$Y'' = \gamma^2 Y$$

The general solution can be written in terms of hyperbolic sine and hyperbolic cosine.

$$Y(y) = C_5 \cosh \gamma y + C_6 \sinh \gamma y$$

Differentiate it with respect to $y$.

$$Y'(y) = \gamma(C_5 \sinh \gamma y + C_6 \cosh \gamma y)$$

Apply the boundary conditions to determine $C_5$ and $C_6$.

$$\begin{aligned} Y'(0) &= \gamma(C_6) = 0 \\ Y'(H) &= \gamma(C_5 \sinh \gamma H + C_6 \cosh \gamma H) = 0 \end{aligned}$$

Since $C_6 = 0$, the second equation reduces to $C_5 \gamma \sinh \gamma H = 0$. No nonzero value of $\gamma$ can satisfy this equation, so $C_5 = 0$.

$$Y(y) = 0$$

This is the trivial solution, so there are no negative eigenvalues. According to the principle of superposition, the solution to the PDE is a linear combination of the eigenfunctions $u = X_n(x)Y_n(x)$ over all the eigenvalues.

$$\boxed{u(x,y) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left(-\frac{n\pi}{H}x\right)\cos\frac{n\pi y}{H}}$$

Differentiate it with respect to $x$.

$$\frac{\partial u}{\partial x} = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\exp\left(-\frac{n\pi}{H}x\right)\cos\frac{n\pi y}{H}$$

Use the final boundary condition to determine the coefficients $A_n$.

$$\frac{\partial u}{\partial x}(0,y) = \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\cos\frac{n\pi y}{H} = f(y)$$

Multiply both sides by $\cos\frac{p\pi y}{H}$.

$$\sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\cos\frac{n\pi y}{H}\cos\frac{p\pi y}{H} = f(y)\cos\frac{p\pi y}{H}$$

Integrate both sides with respect to $y$ from 0 to $H$.

$$\int_0^H \sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\cos\frac{n\pi y}{H}\cos\frac{p\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{p\pi y}{H}\,dy$$

Split up the integral on the left and bring the constants in front.

$$\sum_{n=1}^{\infty} A_n\left(-\frac{n\pi}{H}\right)\int_0^H \cos\frac{n\pi y}{H}\cos\frac{p\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{p\pi y}{H}\,dy$$

Because the cosine functions are orthogonal, this second integral on the left is zero if $n \neq p$. Only if $n = p$ does it yield a nonzero result.

$$A_n\left(-\frac{n\pi}{H}\right)\int_0^H \cos^2\frac{n\pi y}{H}\,dy = \int_0^H f(y)\cos\frac{n\pi y}{H}\,dy$$

Evaluate the integral.

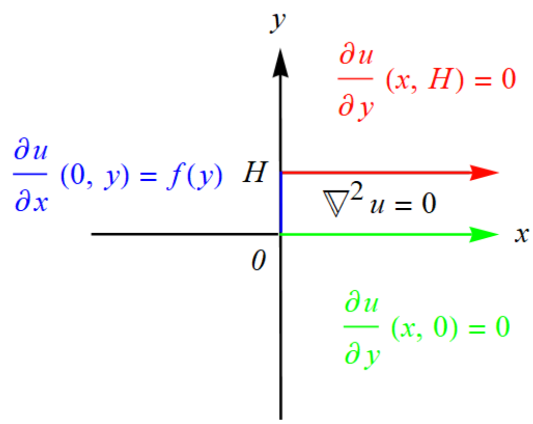
$$A_n\left(-\frac{n\pi}{H}\right)\left(\frac{H}{2}\right) = \int_0^H f(y)\cos\frac{n\pi y}{H}\,dy$$

Therefore,

$$\boxed{A_n = -\frac{2}{n\pi}\int_0^H f(y)\cos\frac{n\pi y}{H}\,dy.}$$

Note that $A_0$ remains arbitrary.

The solvability condition will now be obtained.




$$\nabla^2 u = 0$$

Integrate both sides of the Laplace equation over the semi-infinite rectangular strip of thickness $H$.

$$\iint_R \nabla^2 u \, dA = 0$$

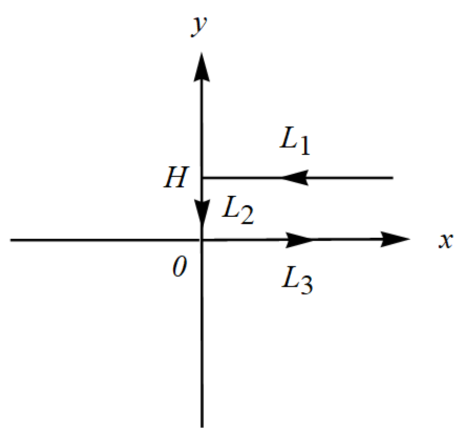
Rewrite the integrand.

$$\iint_R \nabla \cdot \nabla u \, dA = 0$$

Apply the two-dimensional divergence theorem to turn this double integral into a counterclockwise closed loop integral around the boundary.

$$\oint_L \nabla u \cdot \hat{\mathbf{n}} \, ds = 0$$

Here $\hat{\mathbf{n}}$ is the outward unit vector normal to the boundary.




Split up the line integral into the three segments shown above, $L_1$, $L_2$, and $L_3$. The outward unit vectors normal to these segments are $\hat{\mathbf{n}} = \hat{\mathbf{y}}$, $\hat{\mathbf{n}} = -\hat{\mathbf{x}}$, and $\hat{\mathbf{n}} = -\hat{\mathbf{y}}$, respectively.

$$\int_{L_1} \nabla u \cdot \hat{\mathbf{y}} \, ds + \int_{L_2} \nabla u \cdot (-\hat{\mathbf{x}}) \, ds + \int_{L_3} \nabla u \cdot (-\hat{\mathbf{y}}) \, ds = 0$$

Evaluate the dot products.

$$\int_{L_1} \frac{\partial u}{\partial y}\,ds + \int_{L_2} \left(-\frac{\partial u}{\partial x}\right) ds + \int_{L_3} \left(-\frac{\partial u}{\partial y}\right) ds = 0$$

Bring out the minus signs.

$$\int_{L_1} \frac{\partial u}{\partial y}\,ds - \int_{L_2} \frac{\partial u}{\partial x}\,ds - \int_{L_3} \frac{\partial u}{\partial y}\,ds = 0$$

Along $L_1$, $u_y$ is evaluated at $y = H$; along $L_2$, $u_x$ is evaluated at $x = 0$; and along $L_3$, $u_y$ is evaluated at $y = 0$.

$$\int_{L_1} \frac{\partial u}{\partial y}\bigg|_{y=H} ds - \int_{L_2} \frac{\partial u}{\partial x}\bigg|_{x=0} ds - \int_{L_3} \frac{\partial u}{\partial y}\bigg|_{y=0} ds = 0$$

The differential of arc length $ds$ is always positive regardless of whether the path around the boundary is clockwise or counterclockwise. So don't mind the orientation when parameterizing the integration paths.

$$\int_0^\infty \frac{\partial u}{\partial y}\bigg|_{y=H} dx - \int_0^H \frac{\partial u}{\partial x}\bigg|_{x=0} dy - \int_0^\infty \frac{\partial u}{\partial y}\bigg|_{y=0} dx = 0$$

Substitute the prescribed boundary conditions, $u_y(x,0) = 0$ and $u_y(x,H) = 0$ and $u_x(0,y) = f(y)$.

$$\int_0^\infty 0\,dx - \int_0^H f(y)\,dy - \int_0^\infty 0\,dx = 0$$

$$-\int_0^H f(y)\,dy = 0$$

Therefore, multiplying both sides by $-1$,

$$\int_0^H f(y)\,dy = 0.$$

In order for a steady-state solution to exist, this solvability condition must be satisfied. $f$ is not arbitrary.